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On Minimal Interpolating Projections and Trace Duality

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We construct a two-dimensional subspace $V \subset C(K)$ such that an interpolating projection on V is a minimal projection with the norm > 1 . That answers a question posed by B. L. Chalmers. It also answers a question posed implicitly by a theorem of P. Morris and E. W. Cheney. We also give a quantitative generalization of the above mentioned theorem. As is suggested by the title, we use trace duality to obtain these results. © 1991 Academic Press, Inc.

1. INTRODUCTION

This paper addresses a question posed by E. W. Cheney and K. H. Price:

PROBLEM 1.1 [3, Problem 15]. *For what subspaces V in $C(K)$ is it true that at least one of the minimal projections of $C(K)$ onto V is an interpolating projection?*

Here K is compact Hausdorff space and $C(K)$ is the space of all continuous real-valued functions on K .

DEFINITION 1.2. For a finite-dimensional subspace $V \subset C(K)$ we define the projectional constant of V to be

$$\lambda(V) = \inf\{\|P\| : P \text{ projection from } C(K) \text{ onto } V\}.$$

DEFINITION 1.3. We say that a projection P from $C(K)$ onto V is minimal if

$$\|P\| = \lambda(V).$$

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DEFINITION 1.4. A projection P from $C(K)$ onto V is called an interpolating projection if

$$Pf = \sum_{j=1}^n f(k_j)v_j,$$

where $(k_j) \subset K$; $(v_j) \subset V$ is a basis of V and $v_i(k_j) = \delta_{ij}$, $i, j = 1, \dots, n$ and $n = \dim V$.

An easy consequence of a well-known theorem of Nachbin [6] is:

PROPOSITION 1.5 (cf. [4]). *Let V be an n -dimensional subspace of $C(K)$ with $\lambda(V) = 1$. Then there exists an interpolating projection P from $C(K)$ onto V such that P is a minimal projection.*

The following problem was open:

PROBLEM 1.6. *Does the converse to Proposition 1.5 hold?*

In Section 3 of this paper (Theorem 3.1) we construct an example of a subspace $V \subset C(K)$ such that an interpolating projection is a minimal projection onto V yet $\lambda(V) > 1$, hence giving a negative answer to Problem 1.6.

This example also provides a counterexample (cf. Proposition 3.5) to

CONJECTURE 1.7 (Chalmers [1]). *Let V be a finite-dimensional subspace of a Banach space X . Let P be a minimal projection from X onto V . Then P^* is a minimal projection from X^* onto the range of P^* .*

Theorem 4.2 answers Problem 1.1 in the finite-dimensional case.

DEFINITION 1.8. Let P be a projection from $C(K)$ onto V . Define $A_P: K \rightarrow R$ by

$$A_P(k) = \sup\{|(Pf)(k)| : \|f\| \leq 1\}.$$

Clearly $\sup\{|A_P(k)|, k \in K\} = \|P\|$.

In an attempt to solve Problem 1.1, Cheney and Morris proved the following:

THEOREM 1.9. (cf. [2]). *Let V be an n -dimensional Chebyshev subspace of $C(K)$ which admits a minimal interpolating projection P . Then either $\|P\| = 1$ or*

$$\#\{k: A_P(k) = \|P\|\} > n.$$

(Here $\#$ stands for the cardinality of the set).

In Section 4 (Theorem 4.8) we obtain the following generalization of this result:

Let P be a minimal interpolating projection from $C(K)$ onto an n -dimensional Chebyshev subspace V . Then

$$\# \{k: A_P(k) = \|P\| \} > n + (\|P\| - 1)^2.$$

A few words about the methods employed in this paper.

We mostly deal with a finite set K . Hence $C(K) = l_\infty^m$, where $m = \#K$. Let (e_j) be a canonical vector basis in \mathbf{R}_m . If $\dim V = n$ then an interpolation projection from l_∞^m onto V is a projection of the form

$$Px = \sum_{j=1}^n \langle x, e_j \rangle v_j,$$

i.e., the point evaluation functionals in this case are vectors e_j , $j = 1, \dots, n$, considered as the elements of the space l_1^n .

We use an idea that goes at least as far back as Cheney and Price [3] and Cheney and Morris [2] to describe a minimal projection as a solution to a best approximation problem.

Let $\mathcal{L}(l_\infty^m)$ be the space of linear bounded operators from l_∞^m into itself. Let $\mathcal{A} \subset \mathcal{L}(l_\infty^m)$ be a subspace of \mathcal{L} defined by

$$\mathcal{A} := \{A \in \mathcal{L}(l_\infty^m) : \text{Range } A \subset V \subset \ker A\}.$$

Then P is a minimal projection from l_∞^m onto V iff 0 is the best approximation to P from \mathcal{A} . Just as in [2, 3] we use the dual characterization of best approximation to conclude that P is minimal if and only if there exists a functional φ in $[\mathcal{L}(l_\infty^m)]^*$ such that φ annihilates \mathcal{A} (i.e., $\varphi(A) = 0$ for all $A \in \mathcal{A}$) and

$$|\varphi(P)| = \|P\| \|\varphi\|.$$

The relative “novelty” here is the use of trace duality to describe the functionals on $\mathcal{L}(l_\infty^m)$. While the trace duality is frequently used in Banach space theory (cf. [5, 7]) it is especially transparent for $\mathcal{L}(l_\infty^m)$. For convenience we reprove the needed results in Section 2.

In Section 3 we construct a counterexample to Problem 1.6 and Conjecture 1.

In Section 4 we derive a matrix equation which gives a necessary and sufficient condition for a space to have interpolating as a minimal one. As a corollary we derive the generalization of the Cheney–Morris Theorem 2.

2. TRACE DUALITY

In this section we introduce some theoretical aspects of trace duality for the spaces $\mathcal{L}(l_\infty^m)$. Although most of this theory is well-known for general Banach spaces, we find it convenient to prove the results for this particular case emphasizing the specific details needed in the next sections.

We use e_1, \dots, e_m to denote the canonical bases in \mathbf{R}_m , and $\langle \cdot, \cdot \rangle$ to denote the canonical inner product in \mathbf{R}_m . l_∞^m and l_1^m stand for \mathbf{R}_m equipped with the norms

$$\|x\|_\infty = \max\{|\langle x, e_j \rangle|, j = 1, \dots, m\},$$

$$\|x\|_1 = \sum_{j=1}^m |\langle x, e_j \rangle|$$

respectively.

For $y \in l_1^m$ and $v \in l_\infty^m$ we use $y \otimes v$ to define an operator in $\mathcal{L}(l_\infty^m)$ as

$$[y \otimes v](x) := \langle y, x \rangle v.$$

Hence if $A \in \mathcal{L}(l_\infty^m)$ is given by a matrix $A = (a_{ij})$ then

$$A = \sum u_i \otimes e_i,$$

where u_i are the rows vectors of the matrix A . For $A \in \mathcal{L}(l_\infty^m)$ given by a matrix (a_{ij}) we define

$$\|A\| := \max \left\{ \sum_{j=1}^m |a_{ij}|, i = 1, \dots, m \right\}.$$

DEFINITION 2.1. The nuclear norm of $A = (a_{ij})$ is defined by

$$v(A) := \sum_{j=1}^m \max_i |a_{ij}|.$$

Next we will need a somewhat unusual notation.

DEFINITION 2.2. Let $A = (a_{ij}) \in \mathcal{L}(l_\infty^m)$. We use $\tilde{\Sigma}(A)$ to denote the class of matrices (s_{ij}) defined by

$$s_{ij} = \begin{cases} 1 & \text{if } a_{ij} > 0 \\ -1 & \text{if } a_{ij} < 0 \\ \text{any number from } [-1, 1] & \text{if } a_{ij} = 0. \end{cases}$$

DEFINITION 2.3. For a $t \times m$ matrix $A = (a_{ij})$ define an extremal set

$$\mathcal{T}(A) = \left\{ i: \sum_{j=1}^m |a_{ij}| = \|A\|, i = 1, \dots, t \right\}.$$

PROPOSITION 2.4. Let $A = (a_{ij}) \in \mathcal{L}(l_\infty^m)$; $\Gamma = (\gamma_{ji}) \in \mathcal{L}(l_\infty^m)$. Then

1. $|\operatorname{tr}(A\Gamma)| \leq \|A\| \nu(\Gamma)$,
2. The equality in 1 is attained if and only if $\Gamma = \pm [\Sigma(A)]^T D$ where $[\Sigma(A)] \in \tilde{\Sigma}(A)$ and $D = [d_1, \dots, d_m]$ is a diagonal matrix

$$D := \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_m \end{bmatrix}$$

with $d_j \geq 0$ and $d_j = 0$ if $j \notin \mathcal{T}(A)$,

3. In particular for any matrix A there exists Γ such that $\operatorname{tr}(A\Gamma) = \|A\| \nu(\Gamma)$.

Proof. To prove 1, observe that

$$\begin{aligned} |\operatorname{tr} A\Gamma| &= \left| \sum_{i=1}^m \sum_{j=1}^m a_{ij} \gamma_{ji} \right| \leq \sum_{i=1}^m \left(\max_j |\gamma_{ji}| \right) \left(\sum_{j=1}^m |a_{ij}| \right) \\ &\leq \max_i \left(\sum_{j=1}^m |a_{ij}| \right) \cdot \sum_{i=1}^m \max_j |\gamma_{ji}|. \end{aligned} \quad (2.1)$$

To prove 2, note that the first inequality in (2.1) is an equality if and only if each column of the matrix Γ is a positive constant multiple of the vector $(\operatorname{sign} a_{ij}), j = 1, \dots, m$, while the second inequality is an equality if and only if $\gamma_{ji} = 0$ if $j \notin \mathcal{T}(A)$.

Finally 3 follows from 2. ■

PROPOSITION 2.2. Let $N(l_\infty^m)$ stand for the space of all $m \times m$ matrices equipped with the nuclear norm ν . Then $\mathcal{N}(l_\infty^m)$ is isometrically isomorphic to $[\mathcal{L}(l_\infty^m)]^*$.

Moreover, every functional $\varphi \in [\mathcal{L}(l_\infty^m)]^*$ is uniquely identified with a $B \in \mathcal{N}(l_\infty^m)$ by

$$\varphi_B(A) = \operatorname{tr}(AB), \quad \text{for all } A \in \mathcal{L}(l_\infty^m),$$

$$\|\varphi_B\| = \nu(B).$$

Proof. Since $m^2 = \dim \mathcal{L}(l_\infty^m) = \dim [\mathcal{L}(l_\infty^m)]^* = \dim \mathcal{N}(l_\infty^m)$ we con-

clude that the spaces in question are algebraically isomorphic. Proposition 2.4(3) shows that the map $B \rightarrow \varphi_B$ is an isometric isomorphism. ■

Let V_n be an n -dimensional subspace of l_∞^m .

Let $\mathcal{A}(V_n) := \{A \in \mathcal{L}(l_\infty^m) : \text{Range } A \subset V_n \subset \ker A\}$.

Let $\mathcal{P}(V_n) := \{P \in \mathcal{L}(l_\infty^m) : P \text{ is a projection from } l_\infty^m \text{ onto } V_n\}$.

PROPOSITION 2.6. *Let $P \in \mathcal{P}(V_n)$. Then P is minimal if and only if there exist $[\Sigma(P)] \in \tilde{\Sigma}(P)$ and a diagonal operator $D = [d_1, \dots, d_m]$ such that*

$$\text{tr}([\Sigma(P)]^T D A) = 0, \quad \text{for all } A \in \mathcal{A}(V_n). \quad (2.2)$$

Proof. Note that $P + \mathcal{A}(V_n) = \mathcal{P}(V_n)$, for every $P \in \mathcal{P}(V_n)$. Hence, P is minimal if and only if zero is the best approximation to P from $\mathcal{A}(V_n)$. Therefore P is minimal if and only if zero is the best approximation to P from $\mathcal{A}(V_n)$. Therefore P is minimal if and only if there exists a functional $\varphi \in [\mathcal{L}(l_\infty^m)]^*$ such that $\varphi(P) = \|P\| \|\varphi\|$ and $\varphi(A) = 0$ for all $A \in \mathcal{A}$.

According to Proposition 2.5 this is equivalent to the existence of an operator $\Gamma \in \mathcal{A}^*(l_\infty^m)$ with

- (a) $\text{tr}(P\Gamma) = \|P\| \|\Gamma\|$,
- (b) $\text{tr}(A\Gamma) = 0$ for all $A \in \mathcal{A}(V_n)$.

Using Proposition 2.4(2) we conclude that Γ is of the form $[\Sigma(P)]^T D$. ■

Remark 2.7. The space V_n^- consists of all vectors $x \in l_1^m$ that annihilate V_n , i.e.,

$$V_n^- = \{x \in l_1^m : \langle x, v \rangle = 0 \text{ for all } v \in V_n\}.$$

Hence $\dim V_n^- = m - n$. Let v_1, \dots, v_n be a basis for V_n while f_1, \dots, f_{m-n} is a basis for V_n^- .

Then

$$\mathcal{A}(V_n) = \text{span}\{f_i \otimes v_j; i = 1, \dots, m - n; j = 1, \dots, n\}.$$

Therefore $[\Sigma(A)]^T D$ annihilates $\mathcal{A}(V_n)$ if and only if

$$\text{tr}([\Sigma(A)]^T D [f_i \otimes v_j]) = \langle f_i, [\Sigma(A)]^T D v_j \rangle = 0,$$

for all $i = 1, \dots, m - n; j = 1, \dots, n$.

Remark 2.8. Proposition 2.6 is equivalent to: P is minimal if and only if there exists a $\Gamma \in \mathcal{A}^*(l_\infty^m)$ with

- (a) $\text{tr}(P\Gamma) = \|P\| \|\Gamma\|$,
- (b) $\text{tr}(A\Gamma) = 0$ for all $A \in \mathcal{A}(V_n)$.

Remark 2.9. A proposition similar to Proposition 2.5 can be proved for operators in $\mathcal{L}(l_1^m)$.

In this case the norm of $A = (a_{ij}) \in \mathcal{L}(l_1^m)$ is defined to be

$$\|A\| = \max \left\{ \sum_{i=1}^m |a_{ij}|, j = 1, \dots, m \right\},$$

while the norm in $\mathcal{N}(l_1^m)$ is

$$v(A) = \sum_{i=1}^m \max_j |a_{ij}|$$

and $\text{tr}(AB) \leq \|A\| v(B)$.

3. THE MAIN EXAMPLE

THEOREM 3.1. *There exists a two-dimensional subspace $V_2 \subset l_\infty^6$ such that the interpolating projection $P = e_1 \otimes v_1 + e_2 \otimes v_2$ is minimal and $\|P\| > 1$.*

Proof. Let

$$\begin{aligned} v_1 &= \left(1, 0, \frac{2+\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}, \frac{\sqrt{2}}{4} \right), \\ v_2 &= \left(0, 1, \frac{\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -\frac{2+\sqrt{2}}{4} \right), \\ V_2 &= \text{span}\{v_1, v_2\} \subset l_\infty^6. \end{aligned}$$

Consider the projection $P = e_1 \otimes v_1 + e_2 \otimes v_2$. In matrix form

$$P = \begin{bmatrix} 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ \frac{2+\sqrt{2}}{4}, & \frac{\sqrt{2}}{4}, & 0, & 0, & 0, & 0 \\ \frac{\sqrt{2}}{4}, & \frac{2+\sqrt{2}}{4}, & 0, & 0, & 0, & 0 \\ \frac{2+\sqrt{2}}{4}, & -\frac{\sqrt{2}}{4}, & 0, & 0, & 0, & 0 \\ \frac{\sqrt{2}}{4}, & -\frac{2+\sqrt{2}}{4}, & 0, & 0, & 0, & 0 \end{bmatrix}.$$

Next we describe the space $\mathcal{A}(V_2)$. Note that the vector $f = (x_1, x_2, t_1, t_2, t_3, t_4)$ belongs to V_2^\perp if and only if

$$0 = \langle f, v_1 \rangle = x_1 + \frac{2+\sqrt{2}}{4} t_1 + \frac{\sqrt{2}}{4} t_2 + \frac{2+\sqrt{2}}{4} t_3 + \frac{\sqrt{2}}{4} t_4,$$

$$0 = \langle f, v_2 \rangle = x_2 + \frac{\sqrt{2}}{4} t_1 + \frac{2+\sqrt{2}}{4} t_2 - \frac{\sqrt{2}}{4} t_3 - \frac{2+\sqrt{2}}{4} t_4.$$

Choosing (t_1, t_2, t_3, t_4) to be consecutively $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ we obtain the basis f_1, f_2, f_3, f_4 for V_2^\perp as

$$f_1 = \left(-\frac{2+\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, 1, 0, 0, 0 \right),$$

$$f_2 = \left(-\frac{\sqrt{2}}{4}, -\frac{2+\sqrt{2}}{4}, 0, 1, 0, 0 \right),$$

$$f_3 = \left(-\frac{2+\sqrt{2}}{4}, +\frac{\sqrt{2}}{4}, 0, 0, 1, 0 \right),$$

$$f_4 = \left(-\frac{\sqrt{2}}{4}, +\frac{2+\sqrt{2}}{4}, 0, 0, 0, 1 \right).$$

Now $\mathcal{A}(V_2)$ can be written as a linear span of $A_i^{(j)}$ with

$$A_i^{(j)} = f_j \otimes v_i; \quad j = 1, \dots, 4; i = 1, 2.$$

Next pick the operator Γ to be

$$\Gamma = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}.$$

By Proposition 2.4(2) (or by direct calculation) we have

$$\text{tr}(P\Gamma) = \|P\| \, v(\Gamma) = 4 \times \frac{2+2\sqrt{2}}{4} = 2(1+\sqrt{2}),$$

$$\|P\| = \frac{2+2\sqrt{2}}{4} = \frac{1+\sqrt{2}}{2} > 1.$$

To prove that P is minimal (cf. Remark 2.8) it suffices to prove that

$$0 = \text{tr}(\Gamma A_i^{(j)}) = f_j(\Gamma v_i); \quad i = 1, 2; j = 1, 2, 3, 4.$$

We have

$$\begin{aligned} \Gamma v_1 &= \left(1 + \sqrt{2}, 0, 1 + \frac{3\sqrt{2}}{4}, \frac{2 + \sqrt{2}}{4}, 1 + \frac{3\sqrt{2}}{4}, \frac{2 + \sqrt{2}}{4} \right), \\ \Gamma v_2 &= \left(0, 1 + \sqrt{2}, \frac{2 + \sqrt{2}}{4}, \frac{4 + 3\sqrt{2}}{4}, -\frac{2 + \sqrt{2}}{4}, -\frac{4 + 3\sqrt{2}}{4} \right). \end{aligned}$$

It is now easy to verify that $0 = f_j(\Gamma v_1) = f_j(\Gamma v_2)$; $j = 1, 2, 3, 4$. ■

Remark 3.2. The space V_2 in the previous theorem is in fact a symmetric Banach space in the sense that

$$\|\alpha v_1 + \beta v_2\| = \|\beta v_1 + \alpha v_2\| = \|\alpha v_1 - \beta v_2\| \quad \text{for all } \alpha, \beta \in \mathbf{R}.$$

Remark 3.3. The space V_2 constructed above is a Chebyshev subspace of $C(K)$, where $K = \{1, 2, 3, 4, 5, 6\}$ in the sense that V_2 restricted to any two points is two-dimensional. Equivalently every 2×2 submatrix of the matrix

$$\begin{bmatrix} 1 & 0 & \frac{2 + \sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{2 + \sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 1 & \frac{\sqrt{2}}{4} & \frac{2 + \sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{2 + \sqrt{2}}{4} \end{bmatrix}$$

is invertible.

Remark 3.4. It is possible to show that “six” (the dimension of l_x^6) is the least possible. If V_n is a subspace of l_x^m with $m \leq 5$ that admits a minimal interpolating projection then $\lambda(V_n) = 1$.

PROPOSITION 3.5. *The projection P constructed above provides a counterexample to Conjecture 1, i.e., P^* is not a minimal projection from l_1^m onto the range of P^* .*

Proof. Clearly the range of P^* is $\text{span}\{e_1, e_2\} \subset l_1^6$. Consider the projection Q from l_1^6 onto $\text{span}\{e_1, e_2\}$ defined as

$$Qx = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2.$$

Then $\|Qx\|_1 = |\langle x, e_1 \rangle| + |\langle x, e_2 \rangle| \leq \sum_{j=1}^6 |\langle x, e_j \rangle| = \|x\|_1$. Hence $1 = \|Q\| < \|P^*\|$ since $\|P^*\| = \|P\| > 1$. ■

4. CHENEY-MORRIS THEOREM

In this section we will assume that V_n is an n -dimensional subspace of l_x^{n+k} . Every interpolating projection from l_x^{n+k} onto V_n can be written as

$P = \sum_{j=1}^n e_{k_j} \otimes v_j$, where (k_1, \dots, k_n) is a collection of distinct integers between 1 and $n+k$. In this case it will be convenient for us to consider a permutation of the natural basis of l_∞^{n+k} and of its dual so that the same projection P can be written as a block matrix

$$P = \begin{bmatrix} I_{n \times 2} & 0 \\ B^T & 0 \end{bmatrix},$$

where B is a given $n \times k$ matrix (b_{ij}) .

Hence we will treat (e_{k_j}) as the first n coordinates and write the projection as $\sum e_j \otimes v_j$.

Correspondingly we will think of l_∞^{n+k} as $l_\infty^n \oplus l_\infty^k$ and use the notation $[x, y]$ to denote a vector in l_∞^{n+k} , where $x \in l_\infty^n$ consists of the first n coordinates in the permuted basis $((k_1, \dots, k_n)$ coordinates of the standard basis) and $y \in l_\infty^k$ represent the remaining k coordinates. The space V_n is thus spanned by vectors $v_j = [e_j, b_j]$, where $b_j = (b_{j1}, \dots, b_{jk}) \in l_\infty^k$ are the same as the rows of the matrix B appearing above.

Remark 4.1. If $\|P\| = 1$ then $1, 2, \dots, n \in \mathcal{T}(P)$ and hence

$$\#\mathcal{T}(P) \geq n.$$

THEOREM 4.2. *Let $\|P\| > 1$. Then P is minimal if and only if there exists a $\Sigma(B) \in \tilde{\Sigma}(B)$ that satisfies the equation*

$$B\bar{D}[\Sigma(B)]^T B = B\bar{D}A, \quad (4.1)$$

where $\bar{D} = [d_1, \dots, d_k]$ is a $k \times k$ non-zero diagonal matrix with non-negative entries and with $d_j = 0$ if $j \notin \mathcal{T}(B^T)$ (see Definition 2.3) and $A = (\lambda_{ij})$ is a $k \times k$ matrix with $|\lambda_{ij}| \leq 1$.

Remark 4.3. To illustrate Theorem 4.2 we write the appearing matrices explicitly in the case $n=2, j=3$:

$$v_1 = (1, 0, b_{11}, b_{12}, b_{13}), \quad v_2 = (0, 1, b_{21}, b_{22}, b_{23}),$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{22} & b_2 & b_{23} \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ b_{11} & b_{21} & 0 & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 & 0 \\ b_{13} & b_{23} & 0 & 0 & 0 \end{bmatrix}.$$

Assuming that

$$|b_{11}| + |b_{21}| = |b_{12}| + |b_{22}| = |b_{13}| + |b_{23}| = \|P\| > 1, \quad (4.2)$$

i.e., $\mathcal{T}(P) = \{3, 4, 5\}$,

$$\Gamma = \begin{bmatrix} 0 & 0 & d_1(\text{sign } b_{11}) & d_2(\text{sign } b_{12}) & d_3(\text{sign } b_{13}) \\ 0 & 0 & d_1(\text{sign } b_{21}) & d_2(\text{sign } b_{22}) & d_3(\text{sign } b_{23}) \\ 0 & 0 & d_1 \dot{\lambda}_{11} & d_2 \dot{\lambda}_{21} & d_3 \dot{\lambda}_{31} \\ 0 & 0 & d_1 \dot{\lambda}_{12} & d_2 \dot{\lambda}_{22} & d_3 \dot{\lambda}_{32} \\ 0 & 0 & d_1 \dot{\lambda}_{13} & d_2 \dot{\lambda}_{23} & d_3 \dot{\lambda}_{33} \end{bmatrix}.$$

Then Theorem 4.2 reads: there exist $|\dot{\lambda}_{ij}| \leq 1$, and $d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$, $|d_1| + |d_2| + |d_3| > 0$, such that

$$\begin{aligned} & \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} \text{sign } b_{11} & \text{sign } b_{21} \\ \text{sign } b_{12} & \text{sign } b_{22} \\ \text{sign } b_{13} & \text{sign } b_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} \dot{\lambda}_{11} & \dot{\lambda}_{12} & \dot{\lambda}_{13} \\ \dot{\lambda}_{21} & \dot{\lambda}_{22} & \dot{\lambda}_{23} \\ \dot{\lambda}_{31} & \dot{\lambda}_{32} & \dot{\lambda}_{33} \end{bmatrix}. \end{aligned} \quad (4.3)$$

Proof of Theorem 4.2. Assume P is minimal. Since $\|P\| > 1$, the norm $\|P\|$ is not attained in the first n rows of the matrix P . By Proposition 2.6 we conclude that there exists Γ of the form

$$\Gamma = [\Sigma(B)]^T D$$

that annihilates $\mathcal{A}(V_n)$, where

$$D = \begin{bmatrix} 0 & 0 \\ 0 & \bar{D} \end{bmatrix}.$$

From the form of P we have

$$\Gamma = \begin{bmatrix} 0 & [\Sigma(B)] \bar{D} \\ 0 & A^T \bar{D} \end{bmatrix}. \quad (4.4)$$

Let $f = [x, u] \in V_n^\perp$. Then

$$0 = \langle f, v_j \rangle = \langle x, e_j \rangle + \langle u, b_j \rangle, \quad \text{for all } j = 1, \dots, n.$$

Observe that $b_j = B^T e_j$. Thus $\langle x, e_j \rangle + \langle u, B^T e_j \rangle = 0$ or

$$x = -Bu.$$

Hence $V_n^\perp = \{ [-Bu, u] : u \in I_1^k \}$. The condition

$$\text{tr}(\Gamma \cdot [f \otimes v_j]) = 0, \quad \text{for all } j = 1, \dots, n \text{ and for all } f \in V_n^\perp$$

can be written as

$$f(\Gamma v_j) = 0, \quad \text{for all } j = 1, \dots, n \text{ and for all } f \in V_n^\perp. \quad (4.5)$$

From the form of Γ and v_j we have

$$\Gamma v_j = [(\Sigma(B)) \bar{D} b_j, A^T \bar{D} b_j] = [(\Sigma(B)) \bar{D} B^T e_j, A^T \bar{D} B^T e_j].$$

Now the condition (4.5) reads

$$\langle Bu, [\Sigma(B)] \bar{D} B^T e_j \rangle = \langle u, A^T \bar{D} B^T e_j \rangle$$

or

$$\langle u, B^T [\Sigma(B)] \bar{D} B^T e_j \rangle = \langle u, A^T \bar{D} B^T e_j \rangle.$$

Since this equality holds for every $u \in \mathbf{R}_k$ and for all $j = 1, \dots, n$ we have

$$B^T [\Sigma(B)] \bar{D} B^T = A^T \bar{D} B^T.$$

Transposing this equation we have

$$B \bar{D} [\Sigma(B)]^T B = B \bar{D} A.$$

Since every step above can be reversed, Eq. (4.1) implies that the matrix (4.4) satisfies the conditions of Remark 2.8 and hence the converse is also true. ■

COROLLARY 4.4. *Assume the conditions of Theorem 4.2. Let B_0 be a submatrix of the matrix B consisting of the columns $i_j \in \mathcal{T}(P)$. Then P is minimal if and only if there exists a matrix $\Sigma(B_0) \in \tilde{\Sigma}(B_0)$ such that*

$$B_0 \bar{D}_0 [\Sigma(B_0)]^T B_0 = B_0 \bar{D}_0 A_0,$$

where \bar{D}_0 and A_0 are the appropriate submatrices of the matrices \bar{D} and A .

Proof. Let $D^{1/2} = [\sqrt{d_1}, \dots, \sqrt{d_n}]$. Then $\bar{D} = D^{1/2} \cdot D^{1/2}$.

From (4.1) we have

$$(B D^{1/2})(D^{1/2} [\Sigma(B)]^T) B = B \bar{D} A.$$

Since $d_j = 0$ if $j \notin \mathcal{T}(B^T)$ the non-zero columns of $B D^{1/2}$ coincide with $B_0 \bar{D}_0^{1/2}$ while non-zero rows of $D^{1/2} (\Sigma(B))^T$ coincide with the rows of $D_0^{1/2} (\Sigma(B_0))^T$ [cf. (4.3)]. Hence the non-zero columns and rows of the left-

hand side of (4.1) coincide with $B_0 \bar{D}_0 [\Sigma(B_0)]^T B_0$. Similarly, the same holds for the right-hand side.

DEFINITION 4.5. We say that the space $V_n \subset l_{\infty}^{n+k}$ is Chebyshev if the $\dim(V_n | i_1, \dots, i_n) = n$ for any integers $i_1 < i_2 < \dots < i_n \leq n+k$.

In other words, every $n \times n$ submatrix of the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & b_{11} & \cdots & b_{1k} \\ 0 & 1 & \cdots & 0 & b_{21} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n1} & \cdots & b_{nk} \end{bmatrix}$$

is invertible.

Next we will need two simple lemmas.

LEMMA 4.6. Let E be a subspace at l_1^m . Then there exists a projection Q from l_1^m into l_1^m such that $\ker Q = E$ and $\|Q\| < \sqrt{\dim E + 1}$.

Proof. By [5] there exists a projection R from l_1^m onto E with $\|R\| < \sqrt{\dim E}$.

Let $Q = I - R$. Then $\ker Q = \text{range } R = E$ and $\|Q\| \leq 1 + \|R\| < \sqrt{\dim E + 1}$.

LEMMA 4.7 (cf. [8]). Let T be a right invertible operator from l_1^m onto l_1^q . Then the set of all projection $Q \in \mathcal{L}(l_1^m)$ such that

$$\ker Q = \ker T$$

coincides with the set

$$\{ST : TS = I\}.$$

Proof. Trivially $TS = I$ implies that ST is a projection in $\mathcal{L}(l_1^m)$. Since S is left invertible we have $\ker S = \{0\}$ and

$$\ker ST = \ker T.$$

Conversely let $Q \in \mathcal{L}(l_1^m)$ be a projection with $\ker Q = \ker A$. Let $V_0 = \text{range } Q$.

Then $\dim V_0 = q$ and $l_{\infty}^m = \ker Q \otimes V_0$. Define an operator T_0 by restricting T to V_0 :

$$T_0 = T|_{V_0} : V_0 \rightarrow l_1^q.$$

Then $\ker T_0 = \{0\}$; T_0 maps a q -dimensional space into q -dimensional

space. Hence T_0 is invertible. Clearly $S = T_0^{-1}Q$ is the right inverse to T and

$$Q = ST. \quad \blacksquare$$

We are now ready to prove the generalization of the Cheney–Morris Theorem:

THEOREM 4.8. *Let V_n be a Chebyshev subspace of l_∞^{n+k} . Let P be an interpolating projection from l_∞^{n+k} onto V_n such that $\|P\| = \lambda(V_n) > 1$. Then*

$$\#\mathcal{T}(P) > n + (\|P\| - 1)^2.$$

Proof. In view of Corollary 4.4 we may assume that $\#\mathcal{T}(P) = k$. Consider two cases. First, let $k \leq n$. Then the matrix B from Theorem 4.2 is left invertible. Hence (from (4.1))

$$\bar{D}[\Sigma(B)]^T B = \bar{D}A.$$

Suppose that $d_j \neq 0$. Then the j -s diagonal entry of $\bar{D}[\Sigma(B)]^T B$ is $d_j \sum_{i=1}^n |b_{ij}| = d_j \|P\|$ while the j -s diagonal entry of $\bar{D}A$ is $d_j \lambda_{ij}$. Hence $d_j \|P\| = d_j \lambda_{ij} = d_j |\lambda_{ij}| \leq d_j$. So $\|P\| \leq 1$ which contradicts the assumption of the theorem.

Now suppose $k > n$. Then $\dim \ker B = k - n$ and B is invertible from the right. By Lemmas 4.6 and 4.7 we can choose a right inverse S such that $\|SB\| < \sqrt{k - n} + 1$. Here $SB: l_1^k \rightarrow l_1^k$. From (4.1) we have

$$\text{tr}(B \bar{D}[\Sigma(B)]^T) = \text{tr}(B \bar{D}AS) = \text{tr}(SB \bar{D}A).$$

By direct computation (cf. (4.3))

$$\text{tr}(B \bar{D}[\Sigma(B)]^T) = (\Sigma d_j) \|P\|.$$

By trace-duality for $\mathcal{L}(l_1^k)$ (cf. Remark 2.9)

$$\text{tr}(SB \bar{D}A) \leq \|SB\| v(\bar{D}A) \leq (\sqrt{k - n} + 1)(\Sigma d_j).$$

Hence

$$\|P\| < (\sqrt{k - n} + 1),$$

which implies

$$k - n > (\|P\| - 1)^2$$

or

$$\#\mathcal{T}(P) = k > n + (\|P\| - 1)^2. \quad \blacksquare$$

Remark 4.9. The proof of Theorem 4.8 strongly depends on the fact that V_n is Chebyshev. Otherwise, we could not conclude the invertibility of the matrix B .

Without this assumption we can show that if $V_n \subset l_\infty^{n+2}$ admits a minimal interpolating projection then $\lambda(V_n) = 1$.

As we mentioned earlier for $n = 2$ we can improve this to $V_2 \subset l_\infty^5$.

Remark 4.10. From (4.1) it is easy to deduce that if P is minimal and the vector v_j has the property $v_{ji}v_{jk} \geq 0$ for all j, i, k then $\|P\| = 1$.

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